A unified convention for achievement positional games

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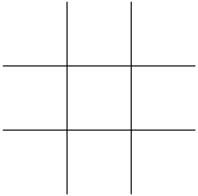
Positional Games (Hales et Jewett 1968)

Two players, **Left** and **Right**, play on a hypergraph $H = (V, E)$ ($E \subset 2^V$).

Left and **Right** take turns picking a previously unpicked vertex.

Hyperedges represent the goals.

"Achievement" $\rightarrow$ Hyperedges represent goals to achieve: winning sets.



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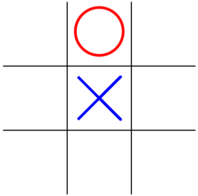

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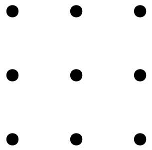

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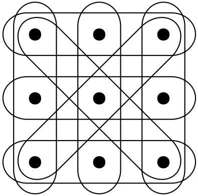


Maker-Maker Convention

Maker-Maker convention = Hyperedges are winning sets for both players (the first player to fill an hyperedge wins).

By strategy-stealing argument, only the first player to play, **Left**, can have a winning strategy.

Question : Does **Left** have a winning strategy ?

Theorem (Koepke 2025 (+ Biskov 2004))

6-*uniform Maker-Maker game is PSPACE-complete.*

Theorem (Trivial)

2-*uniform Maker-Maker game is in LOGSPACE.*

Maker-Breaker Convention

Maker-Breaker convention = Hypergedges are winning sets for **Left** , **Right** wins if **Left** does not win.

Theorem (Schaefer 1978)

11-*uniform Maker-Breaker game is PSPACE-complete.*

Theorem (Koepke 2025)

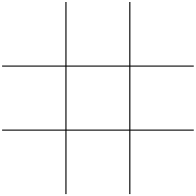
5-*uniform Maker-Breaker game is PSPACE-complete.*

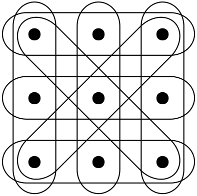
Theorem (Galliot, Gravier, Sivignon 2022)

3-*uniform Maker-Breaker game is polynomial.*

Intermediate position of Maker-Maker games

A Maker-Maker instance





Intermediate position of Maker-Maker games

Not a Maker-Maker instance

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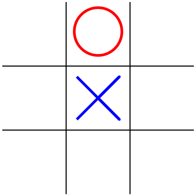
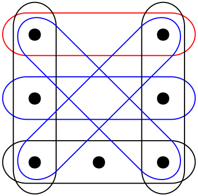
Not a Maker-Maker instance





Two distinct colors of hyperedges, representing winning conditions for **Left** and **Right** (black hyperedges are both red and blue).

Definition

An **Achievement positional game** is a triplet $H = (V, E_L, E_R)$ such that

- (V, E_L) and (V, E_R) are hypergraphs.
- E_L (resp. E_R) are winning conditions for **Left** (resp. **Right**).
- Can be seen as a hyperedge-colored hypergraph with three colors (red, blue, and black = red and blue).

Special cases:

- $E_L = E_R \to$ Maker-Maker games
- $E_R = \emptyset \sim$ Maker-Breaker games

Deciding the outcome

Question : How hard is it to decide whether **Left** has a winning strategy ?

PSPACE-c in general (includes both Maker-Maker and Maker-Breaker conventions).

Natural fragments of interest:

- Bounds on the size for hyperedges in E_L and E_R
  $\rightarrow$ Extension of Maker-Breaker games
- Bound k (resp. $k-1$) on elements of $E_L \cap E_R$ (resp. $E_L \triangle E_R$)
  $\rightarrow$ Intermediate positions of rank k Maker-Maker games
- E_R is a maximal set of minimal transversal of E_L
  $\rightarrow$ Maker-Breaker games ... except size definition

Bounding sizes

Question : Does **Left** has a winning strategy ?
Maximum respective sizes for elements of E_L (in blue) and E_R (in red).

	0 1	2	3	4	5+
0 1	L	L	P	?	PSPACE-c
2	L				PSPACE-c
3+	L				PSPACE-c

First row $\sim$ Maker-Breaker games.

Hyperedges have size at least 1, thus 0 means no hyperedge.

Rows (resp. Columns) 0 and 1 are merged due to trivial reductions.

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	0 1	2	3	4	5+
0 1	L	L	P	?	PSPACE-c
2	L	**P**	**NP-hard**	**NP-hard**	PSPACE-c
3+	L	**co-NP**-c	**PSPACE**-c	**PSPACE**-c	PSPACE-c

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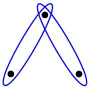
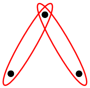
The (2,2) fragment

There is an easy reduction to eliminate hyperedges of size 1.

If every edge has size 2, **Left** wins if and only if there is a blue P_3, since **Right** gets a non-losing pairing strategy otherwise.



If **Left** cannot win, can she prevent **Right** from winning ? If she loses the initiative, **Right** wins if there is a red P_3 left.



The (2,2) fragment

Theorem (Galliot, Sénizergues 2025)

*Computing the outcome (**Left** wins, Draw, or **Right** wins) of an achievement positional game when hyperedges have size at most 2 can be done in time $O(|V|^2(|E_L| + |E_R|)$.*

Sketch of the proof:

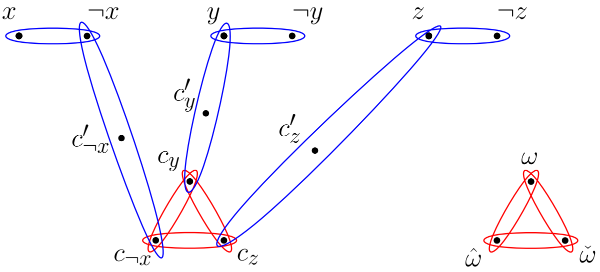
- Classify the sequences of forced moves from a given first move depending on how they end (direct loss, **Left** to play, **Right** to play).
- Playing a sequence of the second type is always optimal for **Left** when possible.
- Computing the maximal forced sequence of x in a given configuration takes time $O(|E_L| + |E_R|)$.

The (2,k) fragment is co-NP-c

In co-NP? **Left** must destroy every red P_3 before losing the initiative to draw the game. Certificate: the sequence of (forcing) moves, easy to check.

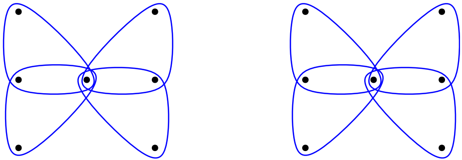
co-NP-hard? Reduction from 3-SAT to the complementary problem.

For a clause $c = \neg x \vee y \vee z$, gadget as follows:



NP-hardness (k,2)

Adding blue butterflies to transform draw into victory for **Left**.



Sketch of the proof: when **Left** destroys the last red P_3, **Right** can play on one of the butterflies. **Left** plays the center of the other and wins since **Right** can only destroy one of the resulting blue P_3 before losing the initiative.

The (3,3) fragment is PSPACE-complete

Theorem (Galliot, Sénizergues 2025)

Deciding whether **Left** *(resp.* **Right***) has a winning strategy is PSPACE-complete when both players have hyperedges of size at most* 3*.*

(Proof by reduction from Quantified Boolean Formula (QBF).)

Corollary

4-uniform Maker-Maker with pre-attributed vertices is PSPACE-complete.

Bounding E_L and E_R: a room for improvement?

	0 1	2	3	4	5+
0 1	L	L	P	?	PSPACE-c
2	L	**P**	**NP-hard**	**NP-hard**	PSPACE-c
3+	L	**co-NP**-c	**PSPACE**-c	**PSPACE**-c	PSPACE-c

Bounding E_L and E_R: a room for improvement?

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0 1	L	L	P	?	PSPACE-c
2	L	**P**	**PSPACE**-c	**PSPACE**-c	PSPACE-c
3+	L	**co-NP**-c	**PSPACE**-c	**PSPACE**-c	PSPACE-c

Theorem (Galliot, Sénizergues 2025+)

Deciding whether **Left** *has a winning strategy is PSPACE-complete even when vertices in* E_L *and* E_R *are of respective maximum sizes* 3 *and* 2.

About Maker-Maker

For Maker-Maker, ranks 3, 4 and 5 are open.

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Theorem (Galliot, Sénizergues 2025+)

Rank 4 Maker-Maker games is PSPACE-complete

Uniformity for ranks 4 and 5?

Is Maker-Breaker that hard really?

When E_R is a maximal set of minimal transversal of E_L $\rightarrow$ Maker-Breaker games ... except size definition

Do Maker-Breaker stays hard when the size of the transversal is included?

Thank you for your attention